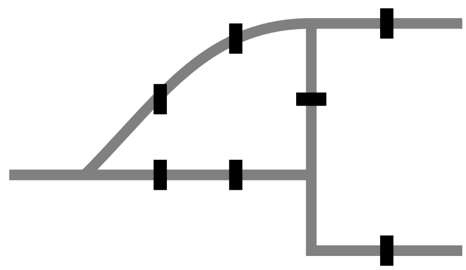
Chapter 1 - Eulerian Graphs

Historical problem: Take a walk in Königsberg and traverse every bridge exactly once. Bridges are black, rivers are gray.

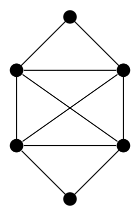


1: Is it possible to traverse every bridge exactly once?

Solution: No

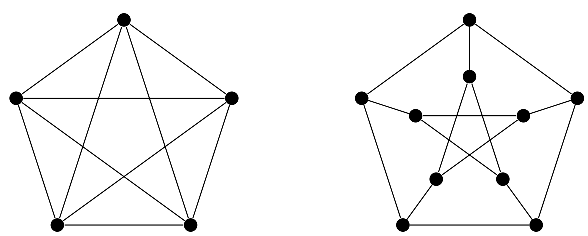
A **trail** is a walk without repeating edges. A **circuit** is a closed trail. A graph is **Eluerian** if it contains a circuit that contains all edges. Such circuit is called an **Eulerian circuit**.

2: Find Eulerian circuit in the following graph



Solution:

3: Decide if K_5 and the Petersen's graph are Eulerian.



Solution: K_5 is and Petersen's is not.

4: Show that if G is Eulerian, then degree of every vertex is even.

Solution: Let v be any vertex of G and C be the circuit. Traverse the circuit and notice that the sequence always looks like eve', where e and e' are edges containing v. Since the circuit is closed, the edges incident to v always come in pairs.

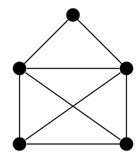
Theorem (Euler 1736) A nontrivial connected graph G is Eulerian if and only if every vertex of G has even degree.

5: Show that if a connected graph G has every vertex of even degree, then G is Eulerian. (Hint: Take longest circuit, induction on number of edges.)

Solution: Use induction on number of edges in G, base case is C_k. In induction step take the longest circuit C. If not all edges covered, there is a vertex x in C that is incident to some uncovered edge. Consider graph $G' = (V(G), E(G) \setminus E(C))$. Notice G' has all vertices of even degree and less edges than G. Take a component of G' that contains x. By induction, it has an Eulerian circuit C'. Notice C' can be inserted in C, which contradicts maximality of C.

An **Eulerian trail** in a graph G is a trail in G containing all edges and does not start and end at the same vertex.

6: Find an Eulerian trail in the following graph



Corollary A connected graph G contains an Eulerian trail if and only if exactly two vertices of G have odd degree. Furthermore, each Eulerian trail of G begins at one of these odd vertices and ends at the other.

7: Prove The Corollary using Euler's theorem

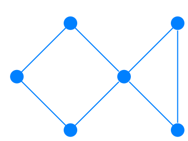
Solution: Let a and b be the two odd vertices. Create a graph G' from G by adding a new edge $e = ab$. This graph has an Eulerian circuit C. Observe that $C - e$ will result in Eulerian trail. (What if ab is already edge in G?)

8: Does every Eulerian bipartite graph have an even number of edges? Explain.

Solution: Yes. Let the graph has partite sets A and B. Partition the edges on the circuit to edges traversed from A to B and from B to A. Since these two kinds alternate, there must be same number of them. This gives that the number of edges is even.

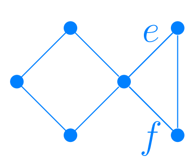
9: Does every Eulerian graph with an even number of vertices have an even number of edges? Explain.

Solution:



10: Prove or disprove the following statement: If G is an Eulerian graph with edges e and f that share a common vertex v, then there is an Eulerian circuit which goes through the edge e and then immediately after through f.

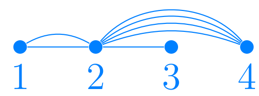
Solution:



A **multigraph** is $G = (V, E)$, where V is the set of vertices and E is a multiset of unordered pairs of vertices.

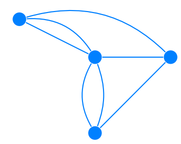
11: Draw the following multigraph $G = (\{1, 2, 3, 4\}, \{\{1, 2\}, \{2, 3\}, \{1, 2\}, \{2, 4\}, \{2, 4\}, \{2, 4\}, \{2, 4\}\})$

Solution:



12: Formulate Königsberg problem as a graph theory problem for Eulerian circuit. Can you do it both as a (simple) graph and a multigraph?

Solution: Normally, it would be a multigraph, but one can do some subdivisions of edges to make it easier. That means, put a new vertex in the middle of every edge.



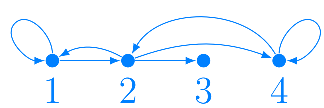
Notice that the trail in some sense has a direction. Hence Eulerian graph can be defined also for graph with "one-way" edges. directed (or oriented) graphs.

A **directed graph** is $D = (V, E)$, where V is the set of vertices and $E \subseteq V \times V$. That means, edges are ordered pairs of vertices, not necessarily distinct.

In a directed graph, edge (v, v) is called a **loop**. For a vertex v, its **out-degree** is $\deg^+(v) = |\{(v, u) \in E\}|$ and **in-degree** is $\deg^-(v) = |\{(u, v) \in E\}|$.

13: Depict the following directed graph $D = (\{1, 2, 3, 4\}, \{(1, 2), (2, 3), (2, 1), (2, 4), (4, 2), (4, 4), (1, 1)\})$. How to indicate the direction of an edge? For all four vertices, write that is their $\deg^+$ and $\deg^-$.

Solution:



A simple graph with orientation on edges is called an **oriented graph**. It has no loop and at most one of $(u, v), (v, u)$. In other words, you take a simple graph and orient the edges.

14: Show that a directed graph D is Eulerian if and only if the graph is connected and at each vertex the in-degree equals the out-degree.

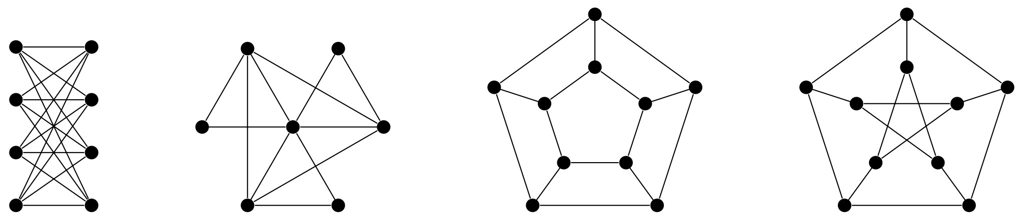
Solution: One can use exactly the same proof as for the undirected case.

15: Prove that if P and Q are paths of maximum length in a connected graph G, then P and Q have at least one vertex in common.

Solution: If not, take any path between any vertex of P and Q and it creates a longer path.

A graph G on n vertices is **Hamiltonian** if it contains a cycle of length n. The cycle is called **Hamiltonian cycle**. (Imagine you want to visit every vertex of a graph once. You don't care about edges.) A path containing all vertices is called **Hamiltonian path**.

16: Decide for the following graphs if they are Hamiltonian, have Hamiltonian path or nothing.



Solution: Both, Path, Both, Path

17 *Open problem*: In a connected graph, pick any three paths of maximum length. Is there always a vertex that lies on all of them?